A marathon through General Relativity and Cosmology

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"Are you ready for Planck?"
a "student/postdoc/faculty/guest speaker/friends..." seminar

TU Dortmund

Cosmology

...is the science of the origin, structure, and space-time relationships of the Universe.

Copernican Principle: There are no “special” observers in the Universe!

The Cosmological Principle: Viewed on sufficiently large distance scales, there are no preferred directions or preferred places in the Universe!

Universe is:

- homogeneous
- isotropic

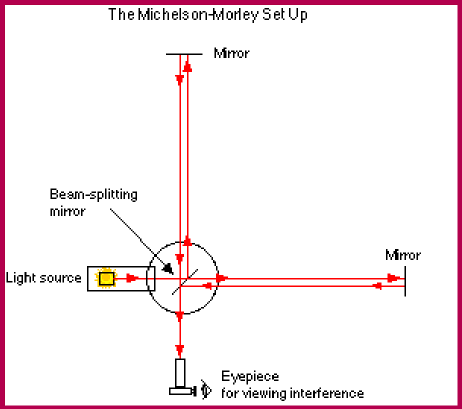
Special and General Relativity in 15 Minutes!

Maxwell's Equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Michelson-Morley experiment



Minkowski space-time

Lorentz Transformation

$$x'^{\alpha} = \Lambda^{\alpha}_{\beta}\, x^{\beta} + a^{\alpha}, \quad \alpha = 0, 1, 2, 3$$

Proper time is invariant:

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = -\eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

General Relativity

Weak Equivalence Principle

The motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of space-time.

Strong Equivalence Principle

The results of any local experiment, gravitational or not, in an inertial frame of reference are independent of where and when in the universe it is conducted.

The metric in the presence of gravitational fields

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

Curvature and Einstein's Equations

Christofell symbol:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

Geodesic Equation:

$$\frac{d^2 x^\mu}{d\lambda^2} - \Gamma^{\mu}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

The Riemann tensor:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}$$

Ricci Tensor and Ricci Scalar (the Curvature Scalar):

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}, \qquad R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

Einstein's field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

Schwarzschild Solution

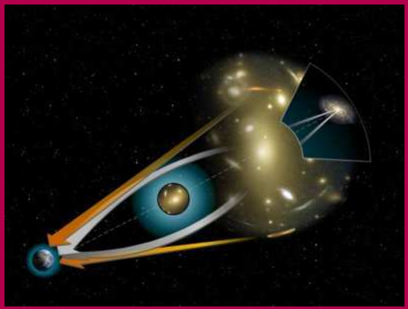
A simple solution of Einstein's field equations:

- spherically symmetric space-time ($\theta \rightarrow -\theta$, $\phi \rightarrow -\phi$)
- static space-time ($\partial g_{\mu\nu}/\partial t = 0$)
- vacuum solution ($T_{\alpha\beta} = 0$)

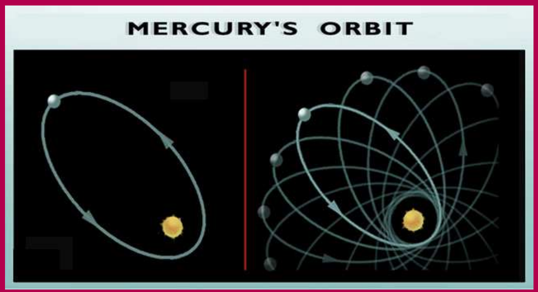
$$ds^2 = -\left[1 - \frac{2MG}{c^2 r}\right] dt^2 + \left[1 - \frac{2MG}{c^2 r}\right]^{-1} dr^2 + r^2 d\Omega^2$$

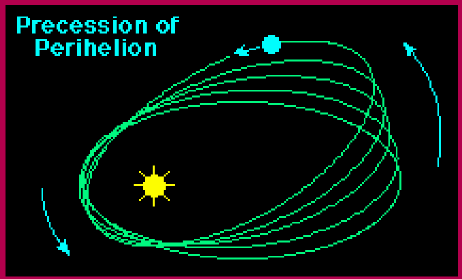
Classical Tests of Einstein's theory

- Deflection of light by the sun/ Gravitational Lensing



- Precession of the perihelia of the orbits of planets





Friedman Robertson Metric

Universe: spatially homogeneous and isotropic which may expand or contract in time. General form of the space-time metric:

$$ds^2 = dt^2 - R(t)^2 \left(d\rho^2 + f(\rho)^2 \left(d\theta^2 + sin^2\theta^2 d\phi^2\right)\right)$$

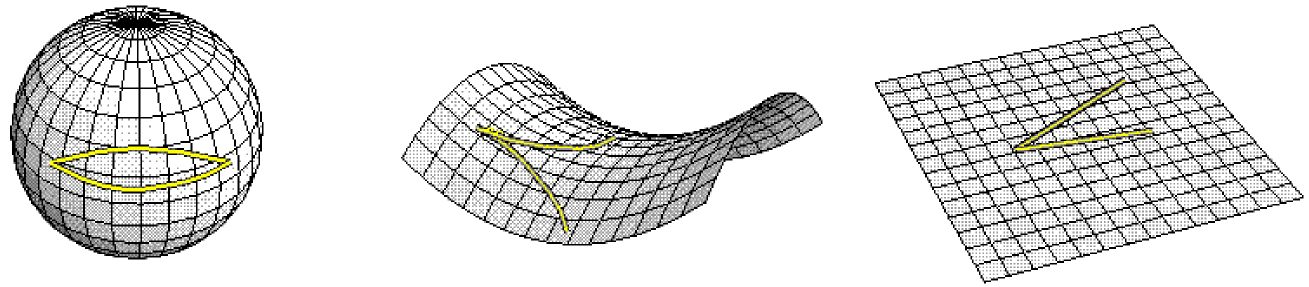
where $f(\rho) = \{sin(\rho), \rho, sinh(\rho)\}$.

After a change of variables FRW metric:

$$d\tau^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 sin^2\theta^2 d\phi^2\right)$$

where

$$\kappa = \begin{cases} +1 & f(\rho) = sin(\rho) \quad \rightarrow \quad positively\ curved\ space \\ 0 & f(\rho) = \rho \quad \rightarrow \quad local\ flatness \\ -1 & f(\rho) = sinh(\rho) \quad \rightarrow \quad negatively\ curved\ space \end{cases}$$



A *closed* universe curves "back on itself". Lines that were diverging apart come back together. Density > critical density.

An *open* universe curves "away from itself". Diverging lines curve at increasing angles away from each other. Density < critical density.

A *flat* universe has no curvature. Diverging lines remain at a constant angle with respect to each other. Density = critical density.

Modelling the Universe

Model of a perfect fluid!!! 4-velocity in comoving coordinates:

$$U^\mu = (1, 0, 0, 0)$$

The energy-momentum tensor:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

Raising one index:

$$T^\mu_\nu = diag\,(-\rho, p, p, p)$$

Equation of state:

$$p = w\rho$$

Conservation of energy equation zero component:

$$
\begin{aligned}
0 &= \nabla_\mu T^\mu_{\ 0} = -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \Rightarrow \\
\frac{\dot{\rho}}{\rho} &= -3(1+w)\frac{\dot{a}}{a} \Rightarrow \\
\rho &\sim a^{-3(1+w)}
\end{aligned}
$$

- matter-dominated $p_M = 0 \ \Rightarrow \ \rho_M \sim a^{-3}$
- radiation dominated $p_R = 1/3\rho_R \ \Rightarrow \ \rho_R \sim a^{-4}$
- vacuum-dominated $p_\Lambda = -\rho_\Lambda \ \Rightarrow \ \rho_\Lambda \sim a^0$

Friedmann Equation

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

Using all $\mu\nu = 00$ and $\mu\nu = ij$ to derive Friedmann Eqs:

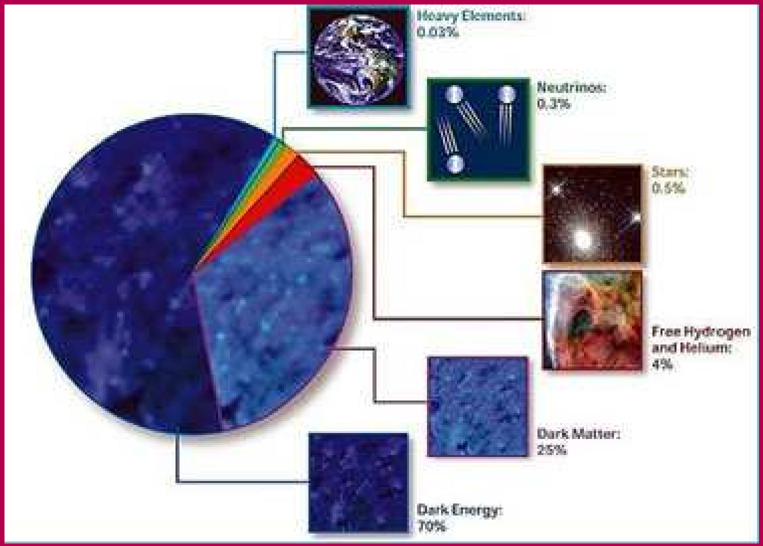
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

- Hubble parameter: $H = \frac{\dot{a}}{a}$, $H_0 \approx 70km/s/Mpc$
- deceleration parameter: $q = -\frac{a\ddot{a}}{\dot{a}^2}$
- density parameter $\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}}$

Density parameter

$$\Omega - 1 = \frac{\kappa}{H^2 a^2}$$

- $\rho < \rho_{crit} \leftrightarrow \Omega < 1 \leftrightarrow \kappa < 0 \leftrightarrow$ open universe
- $\rho = \rho_{crit} \leftrightarrow \Omega = 1 \leftrightarrow \kappa = 0 \leftrightarrow$ flat universe
- $\rho > \rho_{crit} \leftrightarrow \Omega > 1 \leftrightarrow \kappa > 0 \leftrightarrow$ closed universe



Redshifts and Distances

Expanding Universe:

$$\frac{\omega_{obs}}{\omega_{em}} = \frac{a_{obs}}{a_{em}}$$

Redshift:

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \Rightarrow a_{em} = \frac{1}{1+z}$$

Hubble Law:

$$v = H_0 d_P$$

Luminosity distance:

$$d_L^2 = \frac{L}{4\pi F}$$

Evidence for cosmic acceleration: Supernovae type Ia

- Standard candles
- Their intrinsic luminosity is know
- Their apparent luminosity can be measured
- The ratio of the two provides the luminosity-distance d_L
- The red shift z can be measured independently from spectroscopy
- The $d_L(z)$ can be obtained and draw a Hubble diagram

What is the Dark Energy?

- Cosmological Constant

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

- Failure of General Relativity
- Quintessence
- Novel Property of Matter

Cosmological constant?

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

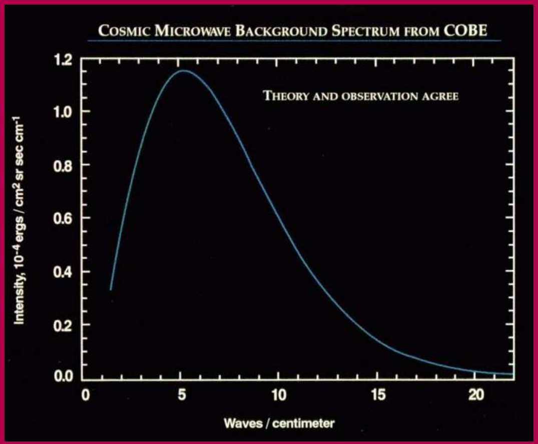
Vacuum energy?

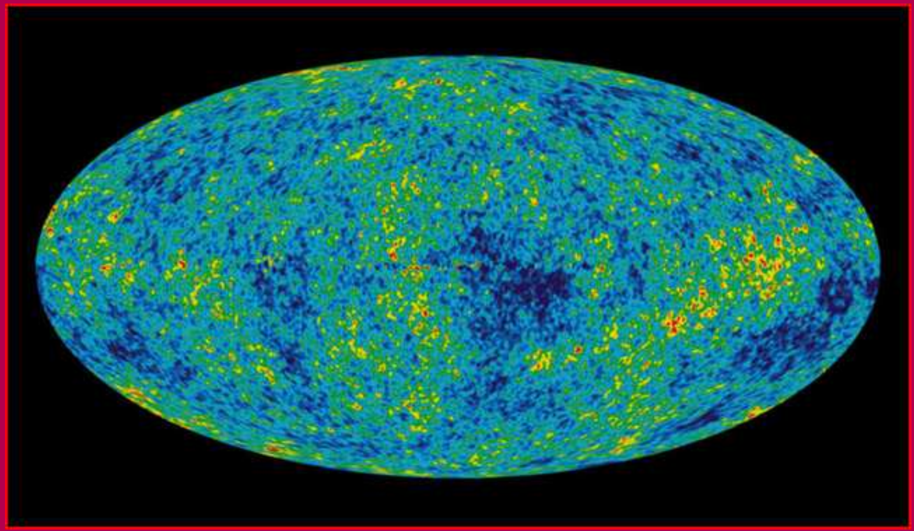
$$\rho_\Lambda \sim \hbar k_{max}^4$$

- $\rho_\Lambda^{Pl} = (10^{18} GeV)^4$
- $\rho_\Lambda^{obs} < (10^{-12} GeV)^4$
  - fantastic cancelation?
  - why is it dominant now? Cosmic coincidence

Cosmic Microvawe Background

- form of electromagnetic radiation
- thermal black body spectrum $(T = 2.725K)$

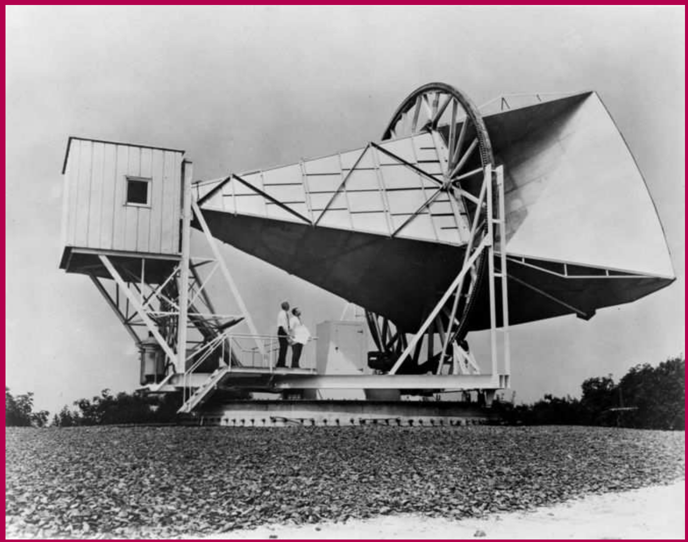




- isotropic

CMBR History

- prediction
  - 1948 by George Gamow and Ralph Alpher
  - R. Alpher and Robert Herman estimated is to 5K
- 1965; Bell Telephone Laboratories; Arno Penzias and Woodrow Wilson discover excess of 3.5 K with horn antenna (1978 Physics Nobel Prize).



What we know so far... for sure?

- Flat, critical density accelerating Universe
- Early period of rapid expansion (inflation)
- Composition: $2/3$ dark energy; $1/3$ dark matter; $1/200$ bright stars
- Matter content: $(29 \pm 4)\%$ cold dark matter; $(4 \pm 1)\%$ baryons; $\geq 0.3\%$ neutrinos
- Current temperature: $T = 2.725 \pm 0.001K$
- Current age: $14 \pm 1Gyr$
- Current expansion rate: $72 \pm 7kms^{-1}Mpc^{-1}$